ATKINSON SCIENCE

THEORY GUIDE

Discharge of a Pressure Vessel
Web Application

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1 Introduction

This report describes the theory underlying the Discharge of a Pressure Vessel web application. The web application calculates the fall in gas pressure in the vessel with time when the valve at the end of the vessel is opened. You can find the application at the web address https://atkinsonscience.co.uk/WebApps/Aerospace/PressureVessel.aspx.

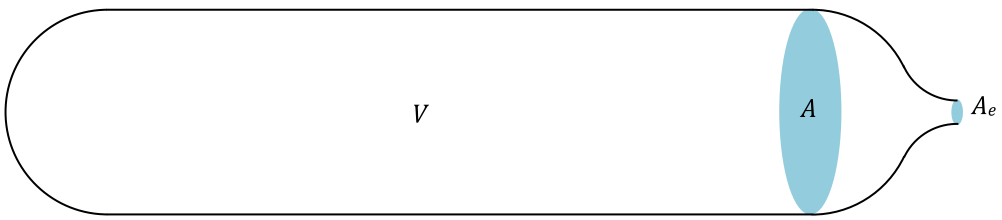
The web application calculates the fall in pressure when (a) there is no heat transfer through the vessel wall to the gas during the discharge (adiabatic discharge), and when (b) the temperature of the gas remains constant (isothermal discharge). The assumption of adiabatic discharge is best suited to the case of very rapid discharge in which there is little time for heat transfer to take place through the vessel wall and the assumption of isothermal discharge is best suited to the case of very slow discharge.

In most cases the pressure in the vessel will be so high that the flow will choke after the valve is opened, then after the pressure has dropped sufficiently the flow will unchoke. The web application calculates the fall in pressure in the vessel during the period of choked flow and during the period of unchoked flow.

1.1 Pressure vessel

The pressure vessel is shown in Figure 1. The vessel has volume V and cross-sectional area A. The end of the vessel tapers to a hole of area A_e through which the gas discharges.

Figure 1 Pressure vessel



1.2 Steady isentropic flow

To analyse the discharge from the pressure vessel, we shall require some results for steady, isentropic flow of a perfect gas. The derivation of these results can be found in any standard textbook on thermodynamics, such as Refs. [1] and [2].

Figure 2 shows flow streamlines in a duct of varying cross-sectional area. We shall assume that the fluid is a perfect gas with constant specific heats and that the flow is steady. Also, we shall assume that the flow is adiabatic and frictionless (isentropic). Under these circumstances, the thermodynamic and flow properties at points 1 and 2 along a streamline are related to each other by the following equations:

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{u_1^2}{2} \qquad (1.1)$$

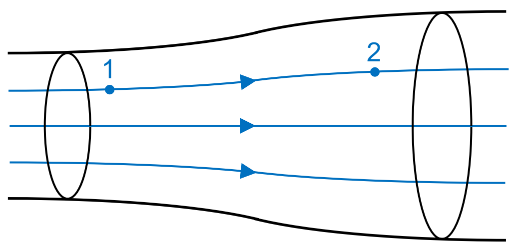
$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{a_2}{a_1}\right)^{\frac{2\gamma}{\gamma-1}} \qquad (1.2)$$

where h [J kg^{-1}] is the specific enthalpy, u [m s^{-1}] is the flow speed, p [Pa] is the pressure, ρ [kg m^{-3}] is the density, T [K] is the temperature, a [m s^{-1}] is the speed of sound, and γ is the ratio of the specific heats of the gas. The speed of sound is given by

$$a^2 = \gamma RT = (\gamma - 1)h \qquad (1.3)$$

where R [J kg^{-1} K^{-1}] is the specific gas constant of the gas.

Figure 2 Flow streamlines in a duct



1.3 Converging nozzle

We shall assume that the end of the vessel through which the gas discharges resembles a converging nozzle. To analyse the discharge of the vessel, we shall require some results for steady, isentropic flow of a perfect gas through a converging nozzle.

If the pressure in the vessel is high enough compared with the pressure beyond the end of the nozzle (the *back pressure* p_b) then the flow leaving the nozzle will reach sonic speed (Mach no. $= 1$) and the flow will be *choked*. During choked flow the rate of discharge depends only on the pressure and temperature upstream of the nozzle and the cross-sectional area of the end of the nozzle A_e and is independent of the back pressure. As the pressure in the vessel falls, the ratio of the back pressure to the gas pressure rises. Once this ratio reaches the *critical ratio* the flow unchokes and the rate of discharge then depends on the back pressure as well as the pressure and temperature upstream of the nozzle.

1.3.1 Choked flow

To analyse the discharge of the pressure vessel, we shall need expressions for the mass flow rate of gas through a converging nozzle during choked flow and unchoked flow. An analysis of steady, isentropic flow of a perfect gas through a converging nozzle can be found in most standard textbooks on thermodynamics for the case of *choked* flow. The mass flow rate is given by

$$\frac{dm}{dt} = \frac{pA_e}{\sqrt{T}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma + 1}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (1.4)$$

and the critical pressure ratio is given by

$$\frac{p_b}{p} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}} \qquad (1.5)$$

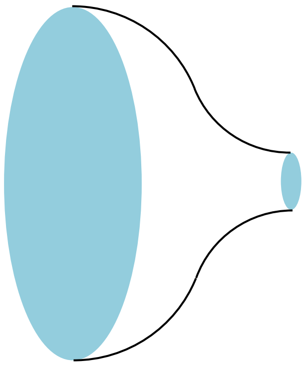
The pressure p and temperature T in (1.4) are the *stagnation* or *reservoir* pressure and temperature in the vessel, respectively, but since the velocity through the area A is very small, we can take them to be the static pressure and static temperature over A. For air $\gamma = 1.4$ and $p_b/p = 0.52828$.

1.3.2 Unchoked flow

The case of *unchoked* flow is rarely covered in standard thermodynamics textbooks. We have therefore provided the necessary analysis in this report.

Figure 3 shows a converging nozzle with inlet area A and exit area A_e. The flow through the nozzle is steady and isentropic with properties p, T, u, etc. at the inlet and p_e, T_e, u_e, etc. at the exit. For unchoked flow we can assume that the exit pressure p_e is equal to the back pressure p_b.

Figure 3 Converging nozzle



The flow through the nozzle is steady, so for continuity the mass flow rate at the exit and the mass flow rate at the inlet must be equal:

$$\dot{m}_e = \dot{m} \qquad (1.6)$$

The mass flow rate at the exit $\dot{m}_e$ is $\rho_e\, u_e\, A_e$ and the mass flow rate at the inlet $\dot{m}$ is $\rho\, u\, A$. Hence we can write (1.6) as

$$\rho_e u_e A_e = \rho u A$$

or

$$\frac{A_e}{A} = \frac{\rho}{\rho_e}\frac{u}{u_e} \qquad (1.7)$$

The flow through the nozzle is assumed to be isentropic, so along the streamlines from the inlet to the exit the flow properties satisfy equations (1.1) to (1.3). From (1.2),

$$\frac{\rho}{\rho_e} = \left(\frac{a}{a_e}\right)^{\frac{2}{\gamma-1}}$$

so (1.7) can be written

$$\frac{A_e}{A} = \left(\frac{a}{a_e}\right)^{\frac{2}{\gamma-1}}\frac{u}{u_e}$$

or

$$u_e^2 = u^2\left(\frac{a}{a_e}\right)^{\frac{4}{\gamma-1}}\left(\frac{A}{A_e}\right)^2 \qquad (1.8)$$

From equation (1.1)

$$h_e + \frac{u_e^2}{2} = h + \frac{u^2}{2}$$

From (1.3), $h = a^2/(\gamma - 1)$, so we can write the preceding equation as

$$\frac{a_e^2}{\gamma - 1} + \frac{u_e^2}{2} = \frac{a^2}{\gamma - 1} + \frac{u^2}{2}$$

We can eliminate u_e from this equation by substituting (1.8):

$$\frac{a_e^2}{\gamma - 1} + \frac{u^2}{2}\left(\frac{a}{a_e}\right)^{\frac{4}{\gamma-1}}\left(\frac{A}{A_e}\right)^2 = \frac{a^2}{\gamma - 1} + \frac{u^2}{2}$$

Multiplying through by $2/a_e^2$ gives

$$\frac{2}{\gamma - 1} + \left(\frac{u}{a_e}\right)^2\left(\frac{a}{a_e}\right)^{\frac{4}{\gamma-1}}\left(\frac{A}{A_e}\right)^2 = \frac{2}{\gamma - 1}\left(\frac{a}{a_e}\right)^2 + \left(\frac{u}{a_e}\right)^2$$

and rearranging this equation gives

$$\left(\frac{u}{a_e}\right)^2 = \frac{\frac{2}{\gamma - 1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{a}{a_e}\right)^2 - 1\right]}{\left(\frac{a}{a_e}\right)^{\frac{4}{\gamma-1}} - \left(\frac{A_e}{A}\right)^2} \qquad (1.9)$$

From equation (1.2),

$$\frac{a}{a_e} = \left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{2\gamma}}$$

Substituting this equation into (1.9) gives

$$\left(\frac{u}{a_e}\right)^2 = \frac{\frac{2}{\gamma - 1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}$$

and taking the square root of both sides gives

$$u = a_e\sqrt{\frac{\frac{2}{\gamma - 1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}}$$

The mass flow rate from the nozzle, $\dot{m} = \rho\, u\, A$, is therefore

$$\dot{m} = \rho A a_e \sqrt{\frac{\frac{2}{\gamma - 1}\left(\frac{A_e}{A}\right)^2 \left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}} \quad (1.10)$$

2 Choked adiabatic discharge

We shall now consider the rate at which the pressure in the vessel falls when the discharge is choked and adiabatic. The gas in the pressure vessel is assumed to be a perfect gas, so at any instant it obeys the equation of state

$$pV = mRT \qquad (2.1)$$

where p [Pa] is the gas pressure, V [m^3] is the internal volume of the vessel, m [kg] is the mass of the gas, R [J kg^{-1} K^{-1}] is the specific gas constant of the gas, and T [K] is the temperature of the gas.

Differentiating (2.1) with respect to time t [s] gives

$$V\frac{dp}{dt} = mR\frac{dT}{dt} + RT\frac{dm}{dt} \qquad (2.2)$$

since V and R are constant. The term dm/dt [kg s^{-1}] is the rate at which the gas is evacuated from the vessel. Before we can integrate (2.2) to find the gas pressure p as a function of time t, we must find expressions for dT/dt and dm/dt in terms of p and t.

$\underline{dT/dt}$

For an isentropic process involving a perfect gas we can apply (1.2):

$$Tp^{(1-\gamma)/\gamma} = C$$

where C is a constant. Taking the natural logarithm of both sides of this equation gives

$$\ln\left[Tp^{(1-\gamma)/\gamma}\right] = \ln C$$

or

$$\ln T + \frac{1-\gamma}{\gamma}\ln p = \ln C$$

Differentiating this equation with respect to t gives

$$\frac{1}{T}\frac{dT}{dt} + \frac{1-\gamma}{\gamma}\frac{1}{p}\frac{dp}{dt} = 0$$

or

$$\frac{dT}{dt} = \frac{\gamma-1}{\gamma}\frac{T}{p}\frac{dp}{dt} \qquad (2.3)$$

$\underline{dm/dt}$

Although the flow from the vessel is unsteady, we shall assume that flow through the exit nozzle can be taken as quasi-steady. We can then use the choked flow equation (1.4) to represent the instantaneous mass flow rate $dm/dt = \dot{m}$:

$$\frac{dm}{dt} = -\frac{pA_e}{\sqrt{T}}\sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

Note that the negative sign appears in front of the right-hand side of this equation because the volume V is *losing* mass

$\underline{dp/dt}$

We can now substitute (2.3) and (1.4) into (2.2) to obtain an expression for dp/dt:

$$V\frac{dp}{dt} = mR\frac{\gamma-1}{\gamma}\frac{T}{p}\frac{dp}{dt} - R\sqrt{T}pA_e\sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (2.4)$$

The term mRT/p in (2.4) is equal to V. Also, from (1.2) we can write

$$T = T_i\left(\frac{p}{p_i}\right)^{(\gamma-1)/\gamma}$$

where T_i and p_i are the initial temperature and pressure in the vessel, respectively. Substituting these relations into (2.4) gives

$$V\left[1-\frac{\gamma-1}{\gamma}\right]\frac{dp}{dt} = -A_e\sqrt{T_i}p\left(\frac{p}{p_i}\right)^{(\gamma-1)/2\gamma}\sqrt{R\gamma\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

or

$$\frac{dp}{dt} = -\frac{\gamma A_e\sqrt{T_i}}{Vp_i^{(\gamma-1)/2\gamma}}\sqrt{R\gamma\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}\,p^{(3\gamma-1)/2\gamma}$$

or

$$\frac{dp}{dt} = -Dp^{(3\gamma-1)/2\gamma} \qquad (2.5)$$

where D is a constant and

$$D = \frac{\gamma A_e\sqrt{T_i}}{Vp_i^{(\gamma-1)/2\gamma}}\sqrt{R\gamma\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (2.6)$$

We can integrate (2.5) to find the gas pressure p as a function of time t:

$$\int_{p_i}^{p} \frac{dp}{p^{(3\gamma-1)/2\gamma}} = -D\int_0^t dt$$

so

$$t = -\frac{2\gamma}{D(1-\gamma)}\left[p^{\left(\frac{1-\gamma}{2\gamma}\right)} - p_i^{\left(\frac{1-\gamma}{2\gamma}\right)}\right]$$

or

$$t = -\frac{2\gamma}{D(1-\gamma)} p_i^{\left(\frac{1-\gamma}{2\gamma}\right)}\left[\left(\frac{p}{p_i}\right)^{\left(\frac{1-\gamma}{2\gamma}\right)} - 1\right] \quad (2.7)$$

Substituting Eq. (2.6) for D into (2.7) gives the following equation for the time t in terms of the gas pressure p:

$$t = \frac{2V\left[\left(\frac{p}{p_i}\right)^{\left(\frac{1-\gamma}{2\gamma}\right)} - 1\right]}{(\gamma-1)A_e\sqrt{T_i}\sqrt{R\gamma\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}} \quad (2.8)$$

We can make this equation dimensionless. The initial speed of sound of the gas a_i is

$$a_i = \sqrt{\gamma R T_i} \quad (2.9)$$

and a characteristic time scale t_{char} is

$$t_{char} = \frac{V}{A_e a_i} \quad (2.10)$$

so we can normalise the time t as follows

$$t^+ = \frac{t}{t_{char}} \quad (2.11)$$

The initial pressure of the gas p_i is a characteristic pressure scale, so the gas pressure can be normalised as follows

$$p^+ = \frac{p}{p_i} \quad (2.12)$$

Equation (2.8) can now be written as

$$\frac{t}{\left(\frac{V}{A_e\sqrt{\gamma R T_i}}\right)} = \frac{\frac{2}{(\gamma-1)}\left[\left(\frac{p}{p_i}\right)^{\left(\frac{1-\gamma}{2\gamma}\right)} - 1\right]}{\sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}}$$

or

$$t^+ = \frac{\frac{2}{(\gamma-1)}\left[(p^+)^{\left(\frac{1-\gamma}{2\gamma}\right)} - 1\right]}{\sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}} \qquad (2.13)$$

so

$$p^+ = \left[1 + \left(\frac{\gamma-1}{2}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} t^+\right]^{\frac{-2\gamma}{(\gamma-1)}} \qquad (2.14)$$

If we substitute the value for air $\gamma = 1.4$ into (2.13) and (2.14), we obtain

$$t^+ = \frac{(p^+)^{-\frac{1}{7}} - 1}{0.11574} \qquad (2.15)$$

and

$$p^+ = [1 + 0.11574 t^+]^{-7}, \qquad (2.16)$$

respectively.

3 Choked isothermal discharge

We shall now consider the rate at which the pressure in the vessel falls when the discharge is choked and isothermal. As before, the gas in the pressure vessel is assumed to be a perfect gas, so at any instant it obeys the equation of state

$$pV = mRT$$

Once again we shall differentiate the equation of state with respect to time, but on this occasion with V, R and T held constant:

$$V\frac{dp}{dt} = RT_i\frac{dm}{dt}$$

or

$$\frac{dp}{dt} = \frac{RT_i}{V}\frac{dm}{dt} \qquad (3.1)$$

As before, we shall assume that the flow through the nozzle can be taken as quasi-steady. Substituting for the mass flow dm/dt from (1.4) gives

$$\frac{dp}{dt} = -\frac{pA_e}{V}\sqrt{\gamma RT_i\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (3.2)$$

Note that the negative sign appears in front of the right-hand side of (3.2) because the volume V is losing mass. The term $\sqrt{\gamma RT_i}$ in (3.2) is the initial speed of sound a_i, so

$$\frac{dp}{dt} = -\frac{pA_e a_i}{V}\sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (3.3)$$

Equation (3.3) can be written

$$\frac{dp}{dt} = -Cp \qquad (3.4)$$

in which the constant C is

$$C = \frac{A_e a_i}{V}\sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (3.5)$$

Integrating (3.4) gives

$$\int_{p_i}^{p}\frac{dp}{p} = -C\int_0^t dt$$

or

$$t = -\frac{1}{C}\ln\frac{p}{p_i} \qquad (3.6)$$

Substituting (3.5) for C into (3.6) gives

$$t = \frac{-V}{A_e a_i \sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}} \ln \frac{p}{p_i} \qquad (3.7)$$

As before, we shall introduce the normalised variables t^+ and p^+:

$$t^+ = \frac{t}{t_{char}}, \qquad t_{char} = \frac{V}{A_e a_i}$$

and

$$p^+ = \frac{p}{p_i}$$

Equation (3.7) can now be written as

$$\frac{t}{\left(\frac{V}{A_e a_i}\right)} = \frac{-1}{\sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}} \ln \frac{p}{p_i}$$

or

$$t^+ = \frac{-1}{\sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}} \ln p^+ \qquad (3.8)$$

so

$$p^+ = \mathrm{Exp}\left[-\left(\frac{\gamma + 1}{2}\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} t^+\right] \qquad (3.9)$$

If we substitute the value for air $\gamma = 1.4$ into (3.8) and (3.9), we obtain

$$t^+ = \frac{-\ln(p^+)}{0.57870} \qquad (3.10)$$

and

$$p^+ = \mathrm{Exp}[-0.57870 t^+], \qquad (3.11)$$

respectively.

4 Unchoked adiabatic discharge

We shall now consider the rate at which the pressure in the vessel falls when the discharge is unchoked and adiabatic. As for the choked discharge, we begin by differentiating the equation of state for a perfect gas with V and R held constant:

$$V\frac{dp}{dt} = mR\frac{dT}{dt} + RT\frac{dm}{dt}$$

As before we shall substitute (2.4) for dT/dt, but now we shall substitute (1.10) for dm/dt because the flow through the nozzle is unchoked:

$$V\frac{dp}{dt} = mR\frac{\gamma-1}{\gamma}\frac{T}{p}\frac{dp}{dt} - RT\rho A a_e\sqrt{\frac{\frac{2}{\gamma-1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}} \qquad (4.1)$$

Note that the negative sign appears before the mass flow rate term in (4.1) because the volume V is losing mass. The term mRT/p in (4.1) is equal to V and the term $RT\rho$ in (4.1) is equal to p, so

$$\frac{dp}{dt} = -\frac{\gamma p A a_e}{V}\sqrt{\frac{\frac{2}{\gamma-1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}} \qquad (4.2)$$

As before, we shall introduce the normalised variables t^+ and p^+:

$$t^+ = \frac{t}{t_{char}}, \qquad t_{char} = \frac{V}{A_e a_i}$$

and

$$p^+ = \frac{p}{p_i}$$

so that equation (4.2) becomes

$$\frac{dp^+}{dt^+} = -\gamma p^+\left(\frac{a_e}{a_i}\right)\sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^+}{p_e^+}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p^+}{p_e^+}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}} \qquad (4.3)$$

From equation (1.2),

$$\frac{a_e}{a_i} = \left(\frac{p_e}{p_i}\right)^{\frac{\gamma-1}{2\gamma}} = \left(p_e^+\right)^{\frac{\gamma-1}{2\gamma}}$$

Substituting into (4.3) gives

$$\frac{dp^+}{dt^+} = -\gamma p^+ (p_e^+)^{\frac{\gamma-1}{2\gamma}} \sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^+}{p_e^+}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p^+}{p_e^+}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}} = f(p^+) \quad (4.4)$$

Integrating equation (4.4) from the time at which the discharge becomes unchoked gives

$$\int_{t_{Unch}^+}^{t^+} dt^+ = \int_{p_{Unch}^+}^{p^+} f(p^+)\, dp^+$$

or

$$t^+ - t_{Unch}^+ = \int_{p_{Unch}^+}^{p^+} f(p^+)\, dp^+ \quad (4.5)$$

This is an equation for time in terms of the gas pressure, rather than vice versa, and the integral on the right-hand side of (4.5) is particularly complex. To make further progress, we decided to integrate (4.4) using a numerical method. An example of a complete calculation of choked adiabatic discharge followed by unchoked adiabatic discharge is given in Section 6.

5 Unchoked isothermal discharge

We shall now consider the rate at which the pressure in the vessel falls when the discharge is unchoked and isothermal. As for the choked discharge, we begin by differentiating the equation of state for a perfect gas with V, R and T constant:

$$\frac{dp}{dt} = \frac{RT_i}{V}\frac{dm}{dt}$$

Substituting (1.10) for dm/dt gives

$$\frac{dp}{dt} = -\frac{RT_i}{V}\rho A a_e \sqrt{\frac{\frac{2}{\gamma-1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A_v}\right)^2}} \qquad (5.1)$$

Note that the negative sign appears before the mass flow rate term in (5.1) because the volume V is *losing* mass. From equation (1.2),

$$\rho = \rho_e\left(\frac{p}{p_e}\right)^{\frac{1}{\gamma}} \qquad (5.2)$$

Substituting (5.2) into (5.1) gives

$$\frac{dp}{dt} = -\frac{RT_i}{V}A a_e \rho_e\left(\frac{p}{p_e}\right)^{\frac{1}{\gamma}} \sqrt{\frac{\frac{2}{\gamma-1}\left(\frac{A_e}{A}\right)^2\left[\left(\frac{p}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p}{p_e}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}} \qquad (5.3)$$

Once again, we shall introduce the normalised variables t^+ and p^+:

$$t^+ = \frac{t}{t_{char}}, \qquad t_{char} = \frac{V}{A_e a_i}$$

and

$$p^+ = \frac{p}{p_i}$$

The normalised form of equation (5.3) is

$$\frac{dp^+}{dt^+} = -\frac{RT_i}{p_i}\left(\frac{a_e}{a_i}\right)\rho_e\left(\frac{p^+}{p_e^+}\right)^{\frac{1}{\gamma}}\sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^+}{p_e^+}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}{\left(\frac{p^+}{p_e^+}\right)^{\frac{2}{\gamma}}-\left(\frac{A_e}{A}\right)^2}} \qquad (5.4)$$

The term RT_i/p_i in (5.4) is equal to $1/\rho_i$, and from equation (1.2),

$$\frac{a_e}{a_i} = \left(\frac{p_e}{p_i}\right)^{\frac{\gamma-1}{2\gamma}} = (p_e^+)^{\frac{\gamma-1}{2\gamma}}$$

so

$$\frac{dp^+}{dt^+} = -\left(\frac{\rho_e}{\rho_i}\right)\left(\frac{p^+}{p_e^+}\right)^{\frac{1}{\gamma}}(p_e^+)^{\frac{\gamma-1}{2\gamma}}\sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^+}{p_e^+}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}{\left(\frac{p^+}{p_e^+}\right)^{\frac{2}{\gamma}}-\left(\frac{A_e}{A}\right)^2}} \qquad (5.5)$$

From (1.2),

$$\frac{\rho_e}{\rho_i} = \left(\frac{p_e}{p_i}\right)^{\frac{1}{\gamma}} = (p_e^+)^{\frac{1}{\gamma}}$$

so

$$\frac{dp^+}{dt^+} = -(p^+)^{\frac{1}{\gamma}}(p_e^+)^{\frac{\gamma-1}{2\gamma}}\sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^+}{p_e^+}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}{\left(\frac{p^+}{p_e^+}\right)^{\frac{2}{\gamma}}-\left(\frac{A_e}{A}\right)^2}} = g(p^+) \qquad (5.6)$$

Integrating equation (5.6) from the time at which the discharge becomes unchoked gives

$$\int_{t_{Unch}^+}^{t^+} dt^+ = \int_{p_{Unch}^+}^{p^+} g(p^+)\,dp^+$$

or

$$t^+ - t_{Unch}^+ = \int_{p_{Unch}^+}^{p^+} g(p^+)\,dp^+ \qquad (5.7)$$

As in the case of adiabatic discharge, we have an equation for time in terms of the gas pressure, rather than vice versa, and the integral on the right-hand side of (5.7) is particularly complex. As before, we opted to use numerical integration. An example of a complete calculation of choked isothermal discharge followed by unchoked isothermal discharge is given in Section 7.

6 Worked example for adiabatic discharge

An insulated cylindrical pressure vessel with a volume V of 0.05 m^3 and an inner diameter D of 0.22 m contains air at an absolute pressure of 10 bar. The valve of the pressure vessel is opened allowing the air to escape to the atmosphere through a hole of diameter D_e of 5 mm. Calculate the time taken for the pressure vessel to discharge completely.

6.1 Choked flow

The cross-sectional area of the pressure vessel is

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 0.22^2}{4} = 0.03801 \text{ m}^2$$

and the cross-sectional area at the end of the nozzle is

$$A_e = \frac{\pi D_e^2}{4} = \frac{\pi \times 0.005^2}{4} = 19.635 \times 10^{-6} \text{ m}^2$$

For air the critical pressure ratio p_b/p is 0.52828. The back pressure p_b is the atmospheric pressure and is equal to 1.01325 bar. Consequently, the discharge from the vessel will unchoke when the gas pressure falls to 1.01325 bar ÷ 0.52828 = 1.9180 bar. When normalised, this pressure is

$$p_{Unch}^+ = \frac{p_{Unch}}{p_i} = \frac{1.9180}{10} = 0.19180$$

The pressure vessel is insulated, so we can assume that the discharge is adiabatic. The gas is air, so the gas pressure during choked adiabatic discharge is given by equations (2.15) and (2.16):

$$t^+ = \frac{(p^+)^{-\frac{1}{7}} - 1}{0.11574}$$

$$p^+ = [1 + 0.11574t^+]^{-7}$$

The value of t^+ at which the discharge unchokes is therefore

$$t_{Unch}^+ = \frac{(p_{Unch}^+)^{-\frac{1}{7}} - 1}{0.11574} = \frac{0.19180^{-\frac{1}{7}} - 1}{0.11574} = 2.2986$$

The speed of sound at the beginning of the discharge is

$$a_i = \sqrt{\gamma R T_i} = \sqrt{1.4 \times 287.055 \times 298.15} = 346.15 \text{ m s}^{-1}$$

and the characteristic timescale is

$$t_{char} = \frac{V}{A_e a_i} = \frac{0.05}{19.635 \times 10^{-6} \times 346.15} = 7.3566 \text{ s}$$

The time at which the discharge unchokes is therefore

$$t_{Unch} = t_{Unch}^+ \times t_{char} = 2.2986 \times 7.3566 = 16.910 \text{ s}$$

In the web application the period 0 to t^{+}_{Unch} is divided into 100 intervals and $p+$ is calculated after each interval. The dimensional values t and p are then calculated and plotted on the user interface so that the user can see how the rate of discharge varies with time. Table 1 shows the first and last few values of t^{+}, p^{+}, t and p.

Table 1 Calculated pressure against time for choked adiabatic discharge

No.	$t+$	$p+$	t [s]	p [bar]
0	0.0	1.0	0.0	10.0
1	0.022986	0.98157	0.16910	9.8157
2	0.045972	0.96353	0.33820	9.6353
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
98	2.2526	0.19755	16.572	1.9755
99	2.2756	0.19465	16.741	1.9465
100	2.2986	0.19180	16.910	1.9180

6.2 Unchoked flow

6.2.1 Fourth-order Runge-Kutta method

We shall use the classical fourth-order Runge-Kutta (RK4) method to integrate equation (4.4) numerically in steps of $h = \Delta t^{+}$ beginning from $(t^{+}_{Unch},\ p^{+}_{Unch})$.

The classical RK4 method is one of several methods that can be used to solve first-order differential equations of the form:

$$\frac{dy}{dx} = f(x, y) \qquad (6.1)$$

There are many variations of the Runge-Kutta method, but in all cases equation (6.1) is cast in the form

$$y_{i+1} = y_i + \emptyset(x_i, y_i, h)h$$

The function $\emptyset(x_i, y_i, h)$ is called an *increment function*. It can be regarded as a representative slope over the interval h. In the classical RK4 method the increment function is

$$\emptyset(x_i, y_i, h) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f(x_i + h, y_i + hk_3)$$

The k's can only be evaluated in the order shown because each k contains all the k's further up the order. When applied to the pressure vessel the RK4 method becomes

$$t_{i+1}^{+} = t_i^{+} + \emptyset(p_i^{+}, h)h$$

$$\emptyset(p_i^{+}, h) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(p_i^{+})$$

$$k_2 = f\left(p_i^{+} + \frac{1}{2}h\right)$$

$$k_3 = f\left(p_i^{+} + \frac{1}{2}h\right)$$

$$k_4 = f(p_i^{+} + h)$$

where f is the right-hand side of (4.4).

6.2.2 Normalised time-step *h*

We can use equation (4.4) to determine dp^{+}/dt^{+} at the start of the period of unchoked flow.

$$\left(\frac{dp^{+}}{dt^{+}}\right)_{Unch} = -\gamma p^{+}(p_e^{+})^{\frac{\gamma-1}{2\gamma}} \sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^{+}}{p_e^{+}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}{\left(\frac{p^{+}}{p_e^{+}}\right)^{\frac{2}{\gamma}} - \left(\frac{A_e}{A}\right)^2}}$$

The normalised exit pressure p_e^{+} is

$$p_e^{+} = p_b^{+} = \frac{p_b}{p_i} = \frac{1.01325}{10} = 0.101325$$

and $p^{+} = p_{Unch}^{+} = 0.19180$, so

$$\left(\frac{dp^+}{dt^+}\right)_{Unch} = -1.4 \times 0.19180 \times 0.101325^{\frac{1}{7}} \sqrt{\frac{5\left[\left(\frac{0.19180}{0.101325}\right)^{\frac{1}{3.5}} - 1\right]}{\left(\frac{0.19180}{0.101325}\right)^{\frac{2}{1.4}} - \left(\frac{19.635 \times 10^{-6}}{0.03801}\right)^2}} = -0.12274$$

We used this pressure gradient to specify the normalised time-step $h = \Delta t^+$ for the period of unchoked flow. During unchoked flow the normalised pressure p^+ falls from p^+_{Unch} to p^+_e. We divided the pressure difference $p^+_e - p^+_{Unch}$ into 100 intervals $\Delta p+$, then divided $\Delta p+$ by $(dp^+/dt^+)_{Unch}$ to obtain h:

$$\Delta p^+ = \frac{p^+_e - p^+_{Unch}}{100} = \frac{0.101325 - 0.19180}{100} = -0.00090475$$

$$h = \Delta t^+ = \frac{\Delta p^+}{(dp^+/dt^+)_{Unch}} = \frac{-0.00090475}{-0.12274} = 0.0073713$$

The $p^+/p^+{}_e$ terms in (4.4) must not be allowed to fall below 1, otherwise the right-hand side of (4.4) will contain the square root of a negative number. To prevent this from happening, the iterations of the RK4 method were only allowed to continue provided that:

$$p^+_{i+1} > p_e - h \times f(p^+_{Unch})$$

where f is the right-hand side of (4.4). Note that h is positive and f is negative. Once this criterion was breeched, the returned value p^+_{i+1} was used to calculate the total normalised discharge time t^+_e from

$$t^+_e = t^+_{i+1} + (p^+_e - p^+_{i+1}) \times \frac{h}{(p^+_{i+1} - p^+_i)}$$

(see Figure 4).

Figure 4 Calculation of the normalised total discharge time

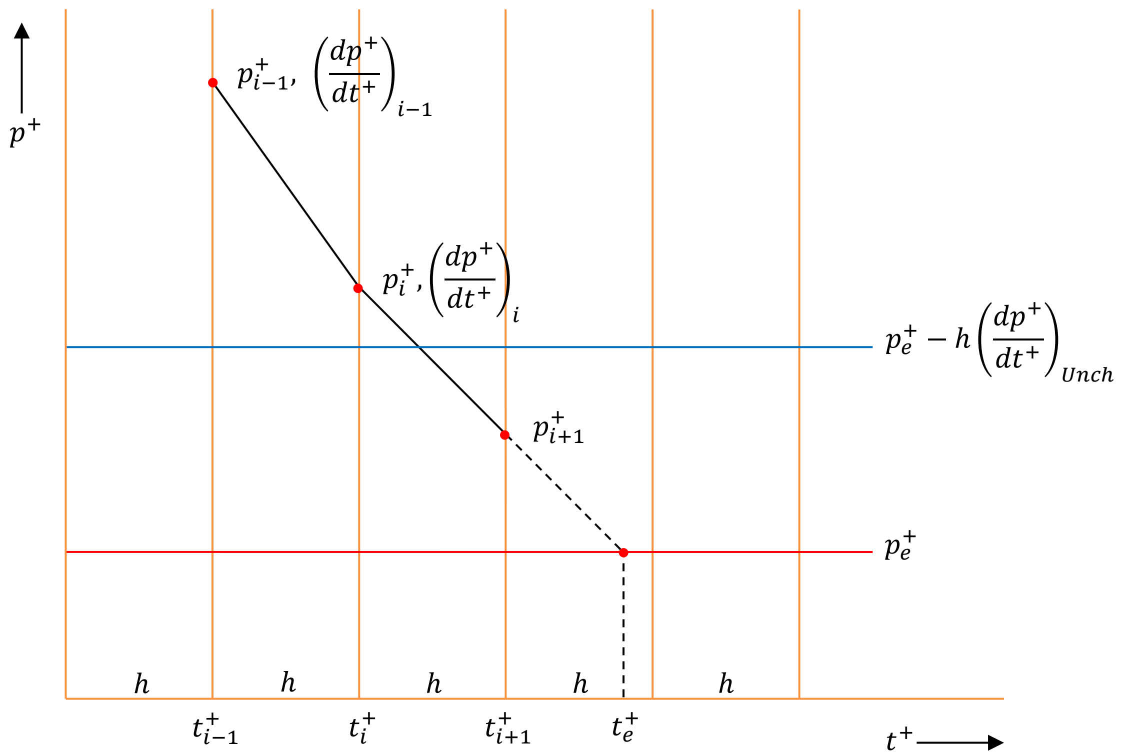


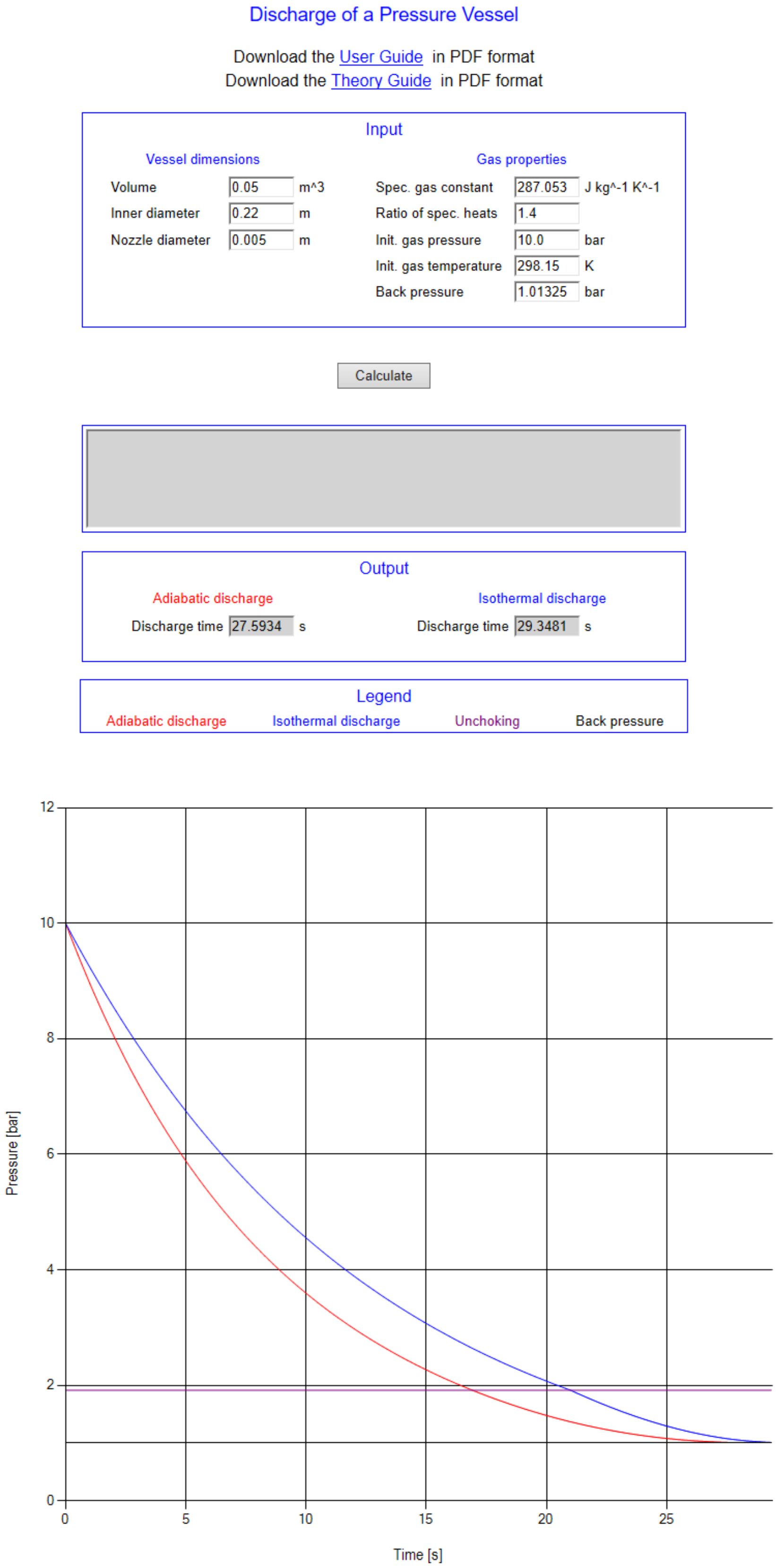
Table 2 shows the first and last few values of t^+, p^+, t and p for unchoked flow. The discharge of the pressure vessel is complete after 27.6 s.

Table 2 Calculated pressure against time for unchoked adiabatic discharge

No.	$t+$	$p+$	t [s]	p [bar]
0	2.2986	0.19180	16.910	1.9180
1	2.3060	0.19090	16.964	1.9090
2	2.3134	0.19000	17.019	1.9000
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
185	3.6624	0.10234	26.942	1.0234
186	3.6697	0.10225	26.997	1.0225
187	3.6771	0.10217	27.051	1.0217
188	3.7508	0.10132	27.593	1.0132

Figure 5 shows the user interface of the web application when it is set up for this example. The fall in pressure when the discharge is adiabatic is given by the red curve in Figure 5. The blue curve gives the fall in pressure when the discharge is isothermal.

Figure 5 User interface for this example



7 Worked example for isothermal discharge

An uninsulated cylindrical pressure vessel with a volume V of 0.05 m³ and an inner diameter D of 0.22 m contains air at an absolute pressure of 10 bar. The valve of the pressure vessel is opened allowing the air to escape to the atmosphere through a hole of diameter D_e of 0.5 mm. Calculate the time taken for the pressure vessel to discharge completely.

7.1 Choked flow

The cross-sectional area of the pressure vessel is

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 0.22^2}{4} = 0.03801 \text{ m}^2$$

and the cross-sectional area at the end of the nozzle is

$$A_e = \frac{\pi D_e^2}{4} = \frac{\pi \times 0.0005^2}{4} = 0.19635 \times 10^{-6} \text{ m}^2$$

For air the critical pressure ratio p_b/p is 0.5283. The back pressure p_b is the atmospheric pressure and is equal to 1.01325 bar. Consequently, the discharge from the vessel will unchoke when the gas pressure falls to 1.01325 bar ÷ 0.52828 = 1.9180 bar. When normalised, this pressure is

$$p_{Unch}^{+} = \frac{p_{Unch}}{p_i} = \frac{1.9180}{10} = 0.19180$$

The pressure vessel is uninsulated and the exit hole is so small that the discharge is likely to be slow. Consequently, we can assume that the discharge is isothermal. The gas is air, so the gas pressure during choked isothermal discharge is given by equations (3.8) and (3.9):

$$t^{+} = \frac{-\ln p^{+}}{0.57870}$$

$$p^{+} = \text{Exp}[-0.57870 t^{+}]$$

The value of t^+ at which the discharge unchokes is therefore

$$t_{Unch}^{+} = \frac{-\ln p_{Unch}^{+}}{0.57870} = \frac{-\ln 0.19180}{0.57870} = 2.8535$$

The speed of sound in the pressure vessel at the beginning of the discharge is the same as for the adiabatic discharge. The characteristic timescale is

$$t_{char} = \frac{V}{A_e a_i} = \frac{0.05}{0.19635 \times 10^{-6} \times 346.15} = 735.66 \text{ s}$$

The time at which the discharge unchokes is therefore

$$t_{Unch} = t_{Unch}^{+} \times t_{char} = 2.8535 \times 735.66 = 2099.2 \text{ s}$$

In the web application the period 0 to t^+_{Unch} is divided into 100 intervals and $p+$ is calculated after each interval. The dimensional values t and p are then calculated and plotted on the user interface so that the user can see how the rate of discharge varies with time. Table 3 shows the first and last few values of t^+, p^+, t and p for the choked isothermal discharge.

Table 3 Calculated pressure against time for choked isothermal discharge

No.	$t+$	$p+$	t [s]	p [bar]
0	0.0	1.0	0.0	10.0
1	0.028635	0.98357	21.065	9.8357
2	0.057269	0.96740	42.130	9.6740
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
98	2.7964	0.19824	2057.2	1.9824
99	2.8249	0.19499	2078.2	1.9499
100	2.8535	0.19180	2099.2	1.9180

7.2 Unchoked flow

We shall use the classical fourth-order Runge-Kutta (RK4) method to integrate equation (5.6) numerically in steps of $h = \Delta t^+$ beginning from (t^+_{Unch}, p^+_{Unch}).

We can use equation (5.6) to determine dp^+/dt^+ at the start of the period of unchoked flow.

$$\left(\frac{dp^+}{dt^+}\right)_{Unch} = -(p^+)^{\frac{1}{\gamma}}(p_e^+)^{\frac{\gamma-1}{2\gamma}}\sqrt{\frac{\frac{2}{\gamma-1}\left[\left(\frac{p^+}{p_e^+}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}{\left(\frac{p^+}{p_e^+}\right)^{\frac{2}{\gamma}}-\left(\frac{A_e}{A}\right)^2}}$$

The normalised exit pressure p_e^+ is

$$p_e^+ = p_b^+ = \frac{p_b}{p_i} = \frac{1.01325}{10} = 0.101325$$

and $p^+ = p^+_{Unch} = 0.19180$, so

$$\left(\frac{dp^+}{dt^+}\right)_{Unch} = -0.19180^{\frac{1}{1.4}} \times 0.101325^{\frac{1}{7}}\sqrt{\frac{5\left[\left(\frac{0.19180}{0.101325}\right)^{\frac{1}{3.5}}-1\right]}{\left(\frac{0.19180}{0.101325}\right)^{\frac{2}{1.4}}-\left(\frac{0.19635\times 10^{-6}}{0.03801}\right)^2}} = -0.14053$$

We used this pressure gradient to specify the normalised time-step $h = \Delta t^+$ for the period of unchoked flow. During unchoked flow the normalised pressure p^+ falls from p^+_{Unch} to p^+_e. We divided this pressure difference into 100 intervals $\Delta p+$, then divided $\Delta p+$ by $(dp^+/dt^+)_{Unch}$ to obtain h:

$$\Delta p^+ = \frac{p_e^+ - p_{Unch}^+}{100} = \frac{0.101325 - 0.19180}{100} = -0.00090475$$

$$h = \Delta t^+ = \frac{\Delta p^+}{(dp^+/dt^+)_{Unch}} = \frac{-0.00090475}{-0.14053} = 0.0064383$$

The normalised total discharge time is calculated in the same way as for the adiabatic flow (see Section 6.2.2).

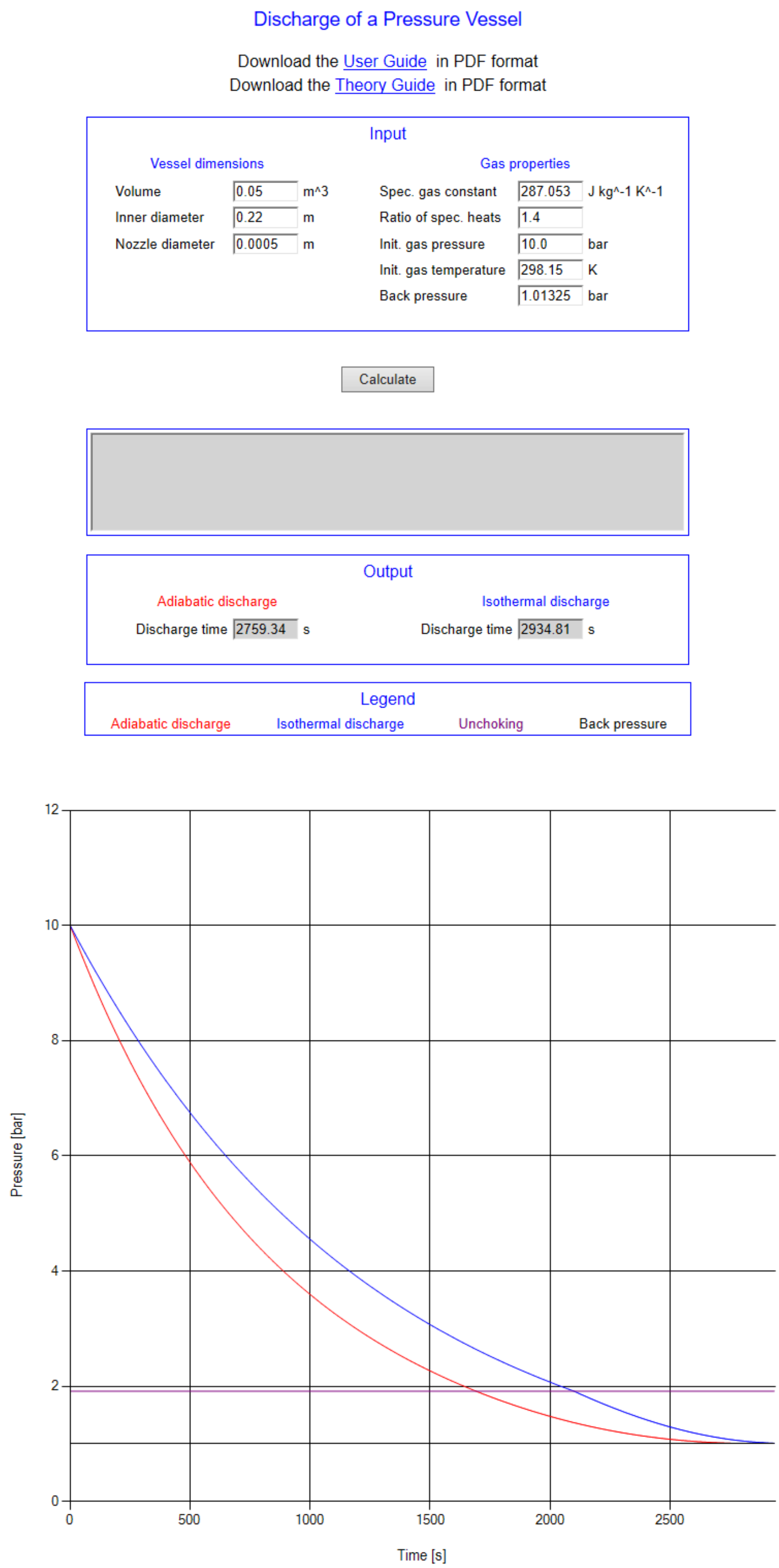
Table 4 shows the first and last few values of t^+, p^+, t and p for unchoked flow. The discharge of the pressure vessel is complete after 2935 s.

Table 4 Calculated pressure against time for unchoked isothermal discharge

No.	$t+$	$p+$	t [s]	p [bar]
0	2.8535	0.19180	2099.2	1.9180
1	2.8599	0.19090	2103.9	1.9090
2	2.8663	0.19000	2108.6	1.9000
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
166	3.9222	0.10240	2885.4	1.0240
167	3.9286	0.10229	2890.2	1.0229
168	3.9351	0.10219	2894.9	1.0219
169	3.9894	0.10132	2934.8	1.0132

Figure 6 shows the user interface of the web application when it is set up for this example. The fall in pressure when the discharge is isothermal is given by the blue curve in Figure 6. The red curve gives the fall in pressure when the discharge is adiabatic.

Figure 6 User interface for this example



8 References

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